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LETTER TO THE EDITOR

Irregular behaviour arising from quasiperiodic forcing of simple quantum systems: insight from perturbation theory

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Abstract. The perturbation of simple quantum systems with almost periodic time-dependent forces can produce irregular dynamics which is manifested in (i) broadband Fourier spectra and (ii) rapid decay of autocorrelation functions. Some classical perturbation methods for differential equations provide insight into this behaviour by showing the existence of a dense spectrum on the real line. Such irregular behaviour is shown also to occur in some model linear systems of differential equations, e.g. a quasiperiodic Mathieu-type equation.

Many studies (Pomeau *et al* 1986, Milonni *et al* 1987, Gerry and Vrscay 1989, to mention only a few) have shown that the quasiperiodic forcing (and kicking) of simple quantum systems can produce an irregular dynamical behaviour of solutions. It is well understood that since the dynamical equations arise from a *linear* Schrödinger evolution equation, chaotic motion in the true (nonlinear) sense of sensitive dependence to initial conditions (SDIC) or positive Lyapunov exponents does not exist. However, a significant irregularity is observed for high field strengths and may be characterized by (i) broadband Fourier spectra and (ii) rapid decay of autocorrelation functions.

In the case of periodic forcing, standard Floquet theory (Hale 1969) establishes the quasiperiodic nature of the solutions. However, it is not readily applicable to quasiperiodic linear systems. It has been shown (Hogg and Huberman 1983) that for a bounded, non-resonant quantum system with a potential $V(x, t)$ which is almost periodic in time, the time evolution of a solution $\psi(x, t)$ to the Schrödinger equation is almost periodic. However, there remains the more important question concerning the nature of the *spectrum* of the solution, which affects its dynamics. For the special case of Fibonacci-type quasiperiodic forcing of a two-level system, the spectrum has been shown to be continuous (Luck *et al* 1988), which is consistent with the irregular dynamics observed.

With particular reference to the two-level spin system studied by Pomeau *et al*, this letter attempts to show that simple perturbation techniques for differential equations can reveal the existence of a continuous Fourier spectrum responsible for the irregular behaviour which is manifested in terms of properties (i) and (ii) given above. The results can, in principle, be generalized to N-level systems. It will be instructive to first examine a simple linear system (example 1)—a harmonic oscillator with quasiperiodic, parametric forcing (Mathieu-type equation)—to show how such forcing of linear systems can indeed lead to highly irregular behaviour. Equations of this kind are relevant to quantum systems since, in the Hamiltonian formulation, all external forcing becomes absorbed into the linear Schrödinger equation to produce a time-dependent system of homogeneous linear differential equations, i.e. $\dot{x}=A(t)x$, rather than a linear system with time-independent inhomogeneous terms, i.e. $\dot{x}=Ax+b(t)$. The only major difference between the classical and quantum systems is that, in the former, the (Euclidean) form $\|x(t)\|$ is not constant. This has inspired example 3 below: a 'toy' classical system for which the above norm is conserved, and which exhibits the irregular behaviour observed for the quantum systems. This letter presents the major results: a detailed analysis of the perturbation treatments will appear elsewhere.

Example 1. Classical example. Consider the following quasiperiodic version of the Mathieu equation for $x(t)\in\mathbb{R}$ (with simplified initial conditions)

$$\ddot{x}+(a^2+\varepsilon\cos t+\varepsilon\cos\chi t)x=0 \qquad x(0)=A \qquad \dot{x}(0)=0 \tag{1}$$

where χ is irrational and, for simplicity, a is assumed rational but non-integer. The parametric modulations are treated as $O(\varepsilon)$ perturbations of a linear oscillator. We consider only those values of (a,ε) which lie away from the classical 'fans' or 'Arnold tongues' of instability associated with parametric excitation (Nayfeh and Mook 1989). (In this case, there are now *two* sets of 'tongues': for $\varepsilon=0$, their boundaries, which characterize 2π- and $2\pi/\chi$-periodic solutions, touch the a axis at the non-negative values $a=n/2$, $m\chi/2$, $m,n=0,1,2,\ldots$. To the author's best knowledge, this feature, which can be generalized to the case of multifrequency quasiperiodic functions in (1), has not been discussed in the literature.) All such solutions of (1) will be bounded and almost periodic (Besicovitch 1954): $x(t)$ admits an expansion of the form

$$x(t)=\sum_{n=0} c_n\,e^{i\lambda_n t}. \tag{2}$$

The set $\Lambda=\overline{\{\lambda_n\}}$ corresponds to the *spectrum* of $x(t)$.

Now, for $\varepsilon\ll 1$, one can first consider the simplest Poincaré-type perturbation expansion (Verhulst 1990) for $x(t)$,

$$x(t)=\sum_{n=0}^{\infty}\varepsilon^n x^{(n)}(t) \qquad x^{(0)}(t)=A\cos at. \tag{3}$$

The corrections $x^{(n)}(t)$ are the solutions to the inhomogeneous second-order differential equations

$$\begin{aligned}&\ddot{x}^{(n)}+a^2x^{(n)}=-[\cos t+\cos\chi t]x^{(n-1)}(t)\\ &x^{(n)}(0)=\dot{x}^{(n)}(0)=0 \qquad n=1,2,\ldots\end{aligned} \tag{4}$$

and have the general form

$$x^{(n)}(t)=\sum_{k=0} A_k^{(n)}(t)\cos(\lambda_k^{(n)}t+\phi_k^{(n)}) \tag{5}$$

where the frequencies are given by the set of all possible non-negative linear combinations $\lambda_k^{(n)}=\{|a\pm k_1\pm k_2\chi|,\ 0\leqslant k_1\leqslant n,\ 0\leqslant k_2\leqslant n\}$. (The earlier assumptions on a avoid the case of *resonance*, where any of the above linear combinations would coincide with a.) Note that secular terms are encountered in this method, i.e. $A_k^{(n)}(t)$ may contain terms in t. This problem can, however, be bypassed with the method of multiple scales (Nayfeh 1973), i.e. define the variables $T_m=\varepsilon^m t$, $m=0,1,2,\ldots$, and consider the expansion

$$x(t)=\sum_{n=0}^{M-1}\varepsilon^m x_m(T_0,\ldots,T_M)+O(\varepsilon T_M). \tag{6}$$

The net result (details to be presented elsewhere): the set of frequencies obtained to all orders in the method consists of the set

$$\{\lambda: \lambda = a \pm k_1 \pm k_2\chi,\ k_1, k_2 \in \{0, 1, 2, \ldots\}\}. \tag{7}$$

The key point is that the multiplication by the cosine terms in (4) may be considered to mimic the 'frequency build-up' of higher harmonics which is characteristic of nonlinear problems (e.g. Duffing oscillator). This would not occur in the case of external forcing. Since χ is irrational, it follows from Kronecker's theorem (Hardy and Wright 1985) that the set of frequencies in (7) is dense on the real line, hence $\Lambda = r\mathbb{R}$, i.e. continuous spectrum. Higher harmonics are damped by appropriate powers ε^n. This is seen in numerically calculated Fourier spectra. As ε increases, however, higher harmonics become discernible and more uniform in magnitude. For $\varepsilon > 1$, the spectrum is quite broadband. As well, the autocorrelation function (defined below) exhibits the same rapid decay as shown for the irregular evolution reported in Pomeau *et al* (1986) and Milonni *et al* (1987). (In this regime, the validity of the perturbation solution breaks down.) In other words, the bounded solutions of a classical quasiperiodic (with incommensurate frequencies) Mathieu equation display *the same irregular behaviour* as observed in the literature for simple quantum systems.

Example 2. The two-level quantum system studied in Pomeau *et al* (1986), which yields Bloch-type equations of motion, can be written in the following matrix form:

$$\dot{\boldsymbol{x}} = [\boldsymbol{A} + \varepsilon \boldsymbol{B}(t)]\boldsymbol{x} \qquad \boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3 \tag{8a}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Omega \\ 0 & \Omega & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & -f(t) & 0 \\ f(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8b}$$

Here, $\Omega = \omega_1 - \omega_2$, where the ω_i are level precession frequencies and $f(t)$ represents the external forcing function. We shall write $f(t) = \cos \chi_1 t + \cos \chi_2 t$, where the χ_i are assumed incommensurate. (This is merely a rewriting of the forcing function $g(t)$ appearing in Pomeau *et al* (1986).) Motion of $x(t)$ is confined to a sphere in the phase space R^3. We now consider again the Poincaré-type perturbation method applied to equation (8):

$$\boldsymbol{x}(t) = \sum_{n=0}^{\infty} \varepsilon^n \boldsymbol{x}^{(n)}(t) \qquad \boldsymbol{x}(0) = \boldsymbol{c} = (c_1, c_2, c_3)^T. \tag{9}$$

The solution to the unperturbed equation ($\varepsilon = 0$) is given by (Hirsch and Smale 1974, chap 5, section 4)

$$\boldsymbol{x}^{(0)}(t) = \mathrm{e}^{tA}\boldsymbol{c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega t & -\sin \Omega t \\ 0 & \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{10}$$

The nth-order perturbation equation is given by the inhomogeneous first-order system

$$\dot{\boldsymbol{x}}^{(n)} = \boldsymbol{A}\boldsymbol{x}^{(n)} + \boldsymbol{B}\boldsymbol{x}^{(n-1)} \qquad \boldsymbol{x}^{(n)}(0) = 0 \qquad n = 1, 2, \ldots \tag{11}$$

with solution (Hirsch and Smale 1974, ch 5, section 5)

$$\boldsymbol{x}^{(n)}(t) = \mathrm{e}^{tA} \int_0^t \mathrm{e}^{-sA} \boldsymbol{B}(s) \boldsymbol{x}^{(n-1)}(s)\, \mathrm{d}s. \tag{12}$$

For this rather simple system, the solutions are easily shown to exhibit the following recursive behaviour for $n = 1, 2, \ldots,$

$$x_1^{(n)}(t) = -\int_0^t f(s) x_2^{(n-1)}(s)\,\mathrm{d}s \tag{13a}$$

$$x_2^{(n)}(t) = \cos\Omega t \int_0^t f(s) x_1^{(n-1)}(s) \cos\Omega s\,\mathrm{d}s + \sin\Omega t \int_0^t f(s) x_1^{(n-1)}(s) \sin\Omega s\,\mathrm{d}s \tag{13b}$$

and an equation similar to (13*b*) for $x_2^{(n)}(t)$. Insertion of (10) into the above, followed by iteration, shows that a 'frequency buildup' similar to that of example 1 operates here. However, secular terms are again encountered. The method of multiple scales may again be applied, to yield that the set of frequencies is given by $\{\lambda\colon \lambda = \pm k_1\Omega \pm k_2\chi_1 \pm k_3\chi_2,\ k_1, k_2, k_3 \in \{0, 1, 2, \ldots\}\}$. This set is again dense on $\mathbb{R}$, hence $\Lambda = \mathbb{R}$, accounting for the irregular behaviour. Higher-level spin systems with external forcing could conceivably be written in the form of (8), and irregular behaviour for sufficiently large field strengths would be expected.

Example 3. A simpler 'toy' problem: It is instructive to consider the following two-dimensional dynamical system,

$$\boldsymbol{A} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & -f(t) \\ f(t) & 0 \end{bmatrix} \tag{14}$$

where $f(t)$ is taken to be the quasiperiodic function used in example 2 above. This 'non-quantal' problem represents a simplification of (8): motion is restricted to a circle of radius $\|\boldsymbol{x}(0)\| = \|\boldsymbol{c}\|$. A perturbation analysis similar to that performed in example 2 shows that the set of frequencies is given by $\{\lambda\colon \lambda = a \pm k_1\chi_1 \pm k_2\chi_2,\ k_1, k_2 \in \{0, 1, 2, \ldots\}\}$, which is dense on $\mathbb{R}$. In this special example, the result may also be derived as follows: if $(x_1(t),\ x_2(t)) = (r\cos\theta(t),\ r\sin\theta(t))$, then $\theta(t) = \theta_0 + at + \int_0^t f(s)\,\mathrm{d}s$. Using the relation (Abramowitz and Stegun 1975)

$$\mathrm{e}^{\mathrm{i}\varepsilon \sin t} = \sum_{k=-\infty}^{k=+\infty} \mathrm{e}^{\mathrm{i}kt} J_k(\varepsilon), \tag{15}$$

where the $J_k(x)$ denote Bessel functions of integer order, the Fourier transform of $\boldsymbol{x}(t)$ may be computed, in agreement with the perturbation result.

In what follows are numerical results for the special case $a = 1$, $\chi_1 = 17711/28657$ and $\chi_2 = 4637/13313$, with $\boldsymbol{x}(0) = (1, 0)$. Following Pomeau *et al* (1986) and Milonni *et al* (1987), these frequencies have been chosen to approximate irrational and incommensurate values. The three cases $\varepsilon = 0.1$, 1.0 and 5.0 are considered here. In particular, the component $x_1(t)$ was studied in terms of the discrete time series $y_k = x_1(kT)$, $k = 0, 1, \ldots,$ where $T = 0.2$ has been chosen. Plots of the power spectra are observed to behave as described in example 1: for $\varepsilon = 0.1$, the dominant 'zeroth order' frequency $a = 1$ is seen, along with the four subdominant first-order frequencies $\omega_k^{(1)}$. For $\varepsilon = 1.0$, many higher harmonics appear at roughly equal amplitude. At $\varepsilon = 5.0$, the spectrum has become very broadband and damped. Figure 1 shows plots of the discrete, normalized time autocorrelation functions (ACF) $C(k)$ corresponding to the y_k, defined as

$$C(k) = \lim_{N\to\infty} \left[\sum_{i=0}^{N} y_i y_{i+k}\right] \left[\sum_{i=0}^{N} y_i y_i\right]^{-1}. \tag{16}$$

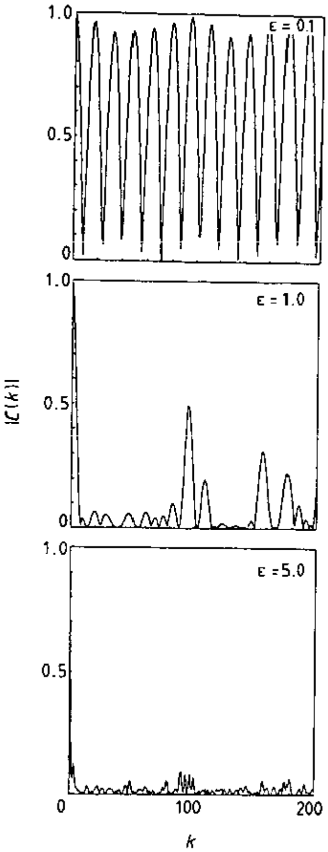

Figure 1. Moduli of the autocorrelation function $C(k)$ in (16) for $\varepsilon = 0.1, 1.0, 5.0$.

(Trivially, $C(0) = 1$. As shown by the spectra, a transition from rather regular motion ($\varepsilon \ll 1$) to highly irregular motion ($\varepsilon > 1$) is demonstrated. Irregularity is characterized by a rapid decrease of the ACF with k, which approximates a white-noise-type signal. (This feature was employed as a potential signature for 'quantum chaos' in the two references cited earlier.) Not surprisingly, the dynamics revealed in these plots is quite similar to that observed for these quantum systems. The general decrease in correlation with increasing ε can also be seen in the behaviour of the quantity $R = \max_{k \geqslant 1} |C(k)|$ for $0 \leqslant \varepsilon \leqslant 10$, plotted in figure 2. There is a great similarity between this graph and figure 5 of Pomeau *et al* (1986). Poincaré (stroboscopic) return maps $\theta_{n+1} = P(\theta_n)$, where $\theta_n = \theta(nT)$, also demonstrate the increasingly irregular behaviour as the perturbaton is strengthened.

The root of the irregular behaviour observed in the simple quasiperiodic linear systems studied above lies in the incommensurate nature of the frequencies composing the forcing term. Perturbation methods reveal a 'frequency build-up' phenomenon analogous to that found in nonlinear equations. This, in turn, accounts for a Fourier spectrum which is dense on the real line. The degree of irregularity increases with the forcing strength, but in a continuous fashion, rather than by a route of bifurcation. It is expected that other quantum systems which may be written in the forms shown above, e.g. N-level spin systems with quasiperiodic forcing, would reveal similar

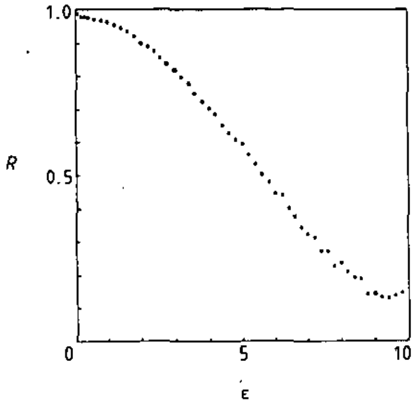

Figure 2. $R = \max_{k\geq 1}|C(k)|$ as a function of the perturbation parameter ε.

irregular behaviour. An examination of infinite dimensional systems, e.g. quasiperiodically perturbed oscillators, is currently in progress.

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